THE TRIANGULAR NUMBERS

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Generating the Triangular Numbers

Start with 1. Add 2 to get 3. Add 3 to get 6. Add 4 to get 10. Continue the pattern to get:

1, 3, 6, 10, 15, 21, 28, 36, 45, …

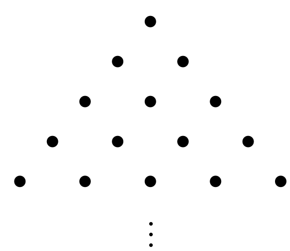
These are called the *triangular numbers*. The formula for producing them is:

$$T_n = T_{n-1} + n$$

where T_n is the nth triangular number.

Knowing that $T_1 = 1$, you can use this formula to find all the triangular numbers, one at a time. This is called the *iterative* method because you take one step, or one iteration, at a time.

The triangular numbers can also be shown this way:

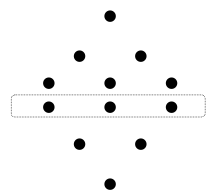


The number of dots in the first 2 rows is the 2nd triangular number, the number of dots in the first 3 rows is the 3rd triangular number, and so on.

If you were asked to find the 15th triangular number, you could use the iterative method or you could extend the triangle to 15 rows and count the dots.

Finding Larger Triangular Numbers

What if you were asked to find the 75th triangular number? Either of the two methods just shown would be too tedious. Luckily, there is a faster way to find triangular numbers.



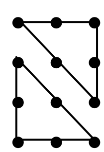
The drawing shows the 3rd triangular number twice, or $2T_3$. It also shows a 3×3 square with 3 extra dots. Therefore, $2T_3 = 3^2 + 3$, or $T_3 = (3^2 + 3)/2$. This applies to every triangular number, so:

$$T_n = (n^2 + n)/2$$

Using this formula makes it easy to find the 75th triangular number:

$$T_{75} = (75^2 + 75)/2 = 2850$$

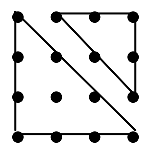
Here is another way to find the formula for the triangular numbers:



Like the previous drawing, this drawing shows the third triangular number twice. It also shows a 3 by 4 rectangle. So, $2T_3 = 3 \times 4$, or more generally, $2T_n = n(n + 1) = n^2 + n$, as before.

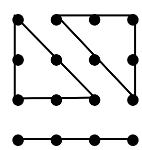
Triangular Numbers and Square Numbers

You can also show that the square numbers are sums of triangular numbers.



The drawing shows that $T_3 + T_4 = 4^2$. More generally, $T_{n-1} + T_n = n^2$. Therefore, every square number is the sum of two consecutive triangular numbers.

Here is another example that relates triangular numbers to square numbers:



This example shows that $2T_3 + 4 = 4^2$. More generally, $2T_{n-1} + n = n^2$. This is equivalent to the previous formula, $T_n = (n^2 + n)/2$. Can you show why?

All these ways of showing the triangular numbers are based on simple groups of dots. There are many other ways of arranging dots to find formulas for the triangular numbers. See if you can discover any other formulas.

Using the Triangular Numbers

You may wonder how anyone could use triangular numbers. While it is true that mathematicians enjoy triangular numbers for their own sake, these numbers also apply to many familiar situations:

Handshakes

Three people are at a meeting. If each person shakes hands exactly once with every other person, 3 handshakes are made (persons 1 & 2, 1 & 3 and 2 & 3). With 4 people, 6 handshakes are made (persons 1 & 2, 1 & 3, 1 & 4, 2 & 3, 2 & 4 and 3 & 4). With n people, T_{n-1} handshakes are made. So 75 people would make $T_{74} = 2775$ handshakes.

Bets

You and your friends are playing poker. The first bet (and "raise") is $1, and each person meets the previous bet and raises by the same amount the last bettor raised plus $1. The second bettor would need to put in $3, the 3rd $6. The 75th bet would total T_{75} = $2850.

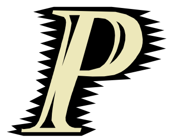
Running

A runner in training runs 5 miles the first day, then 1 mile more each day than the previous day. How many days will it take for the runner to have reached a total of 100 miles?

In addition to running 4 miles each day, the runner runs 1 mile the first day, 2 the second day, and so on. On the second day, the total number of miles run is $(4 \times 2) + (3)$. On the third day, the total is $(4 \times 3) + (6)$. On the nth day, the total is $4n + T_n$. Since $4(10) + T_{10} = 95$ and $4(11) + T_{11} = 110$, the runner reaches 100 miles on the 11th day.

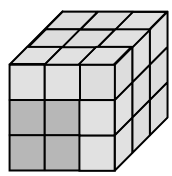
Using the Triangular Numbers (cont.)

Sequences
Consider the sequence A, B, B, C, C, C, D, D, D, D … What is the 75th term?

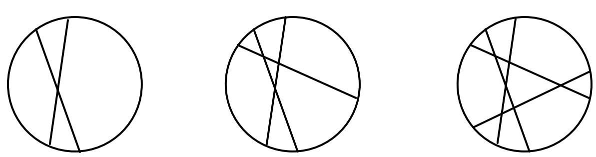


D, the 4th letter, holds the positions from $T_3 + 1$ to T_4. So the nth letter holds the positions from $T_{n-1} + 1$ to T_n. Since $T_{11} = 66$ and $T_{12} = 78$, the 12th letter holds the positions from 67 to 78. Therefore, the 75th term is L.

Cubes
A $3 \times 3 \times 3$ cube is marked into unit cubes. How many cubes of any size could you make using the boundaries of the unit cubes? You can make 1–$3 \times 3 \times 3$ cube, 8–$2 \times 2 \times 2$ cubes and 27–$1 \times 1 \times 1$ cubes for a total of 36 cubes. And you can make 100 cubes from a $4 \times 4 \times 4$ cube. It turns out that you can make T_n^2 cubes from an $n \times n \times n$ cube.



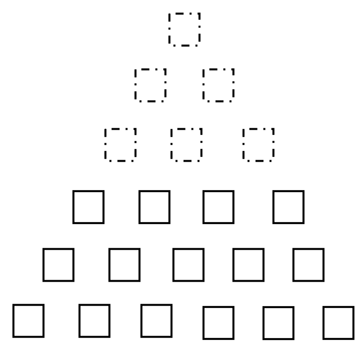
Intersections
Chords of a circle are drawn so that they intersect in the maximum possible number of points. Two chords make 1 intersection point, 3 chords make 3 points, 4 chords make 6 points, etc. The general pattern is that n chords can make a maximum of T_{n-1} intersection points. So 75 chords can make $T_{74} = 2775$ intersection points.



Using the Triangular Numbers (cont.)

Seats at the Movies

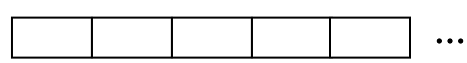
The front row of a movie theater has x seats. Each row has one more seat than the row before. If the last row has y seats, how many seats are in the theater?



Suppose x = 4 and y = 6, as shown above (solid lines). Imagine that the seats continue forward until there is one seat (dotted lines). Then the front has $T_3 = 6$ imaginary seats. If the imaginary seats were actually there, the movie theater would have $T_6 = 21$ seats. Without them, the theater has $T_6 – T_3$ seats = 21 – 6 = 15 seats. In the general case, the theater would have $T_y – T_{x-1}$ seats.

Counting Rectangles

Suppose n congruent rectangles are placed end-to-end to form a long chain, as shown below. Following the borders of any of these rectangles, how many rectangles of any size can be formed?



With just one rectangle in the chain, you can form 1 rectangle. With two rectangles in the chain, you can form two small rectangles and one large rectangle, for a total of 3 rectangles. With three rectangles in the chain you can form 6 rectangles. With n rectangles in the chain, you can form T_n rectangles.

And Many More

Triangular numbers are relevant to every problem involving increases of 1 at each step. Recognizing where these numbers fit into a problem allows describing the solution to the problem with a convenient formula. Search for triangular numbers in the world around you and see where you can find them.

Triangular Numbers and Pascal's Triangle

The first formula given for the triangular numbers, $T_n = T_{n-1} + n$, is not the only iterative formula for determining triangular numbers. Taking differences between consecutive triangular numbers yields other formulas.

Consider three consecutive triangular numbers, T_n, T_{n-1} and T_{n-2}. The difference between the first two numbers in this set of three is:

$$T_n - T_{n-1} = n$$

The difference between the second two numbers is $T_{n-1} - T_{n-2} = n - 1$. The difference between these two differences is:

$$T_n - 2T_{n-1} + T_{n-2} = 1$$

Using this formula requires knowing the two previous triangular numbers. But once you have the first two triangular numbers, the formula can be used to find the rest of them. The method of generating this formula can be repeated. Taking the difference of the differences of the differences yields:

$$T_n - 3T_{n-1} + 3T_{n-2} - T_{n-3} = 0$$

You can extend this method to include as many previous terms as you like. The coefficients of the T_i terms in all these equations are from Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

Each entry in Pascal's triangle is found by adding the entries diagonally above it. Now you can use the 5th row of Pascal's triangle to find the relation between five triangular numbers. From the fourth row of Pascal's triangle down, the sum of the T_i terms with appropriate coefficients is always zero. This shows how Pascal's triangle relates to triangular numbers.

Other Special Numbers
You could explore many other features of triangular numbers, or you could explore other special numbers. Special numbers based on other geometric shapes include:

- Square Numbers
- Pentagonal, Hexagonal, and other Polygonal Numbers
- Centered Polygonal Numbers
- Cubes
- Tetrahedral, Hexahedral, and other Polyhedral Numbers
- Centered Polyhedral Numbers
- Square Pyramidal Numbers

The tetrahedral numbers can be visualized as stacks of triangular numbers. Actually, all the geometric numbers can be expressed in terms of triangular numbers. You may want to try finding some of these expressions.

Other kinds of special numbers include:

- Fibonacci Numbers
- Hailstone Numbers
- Lucas Numbers
- Perfect Numbers
- Prime Numbers
- Star Numbers

There are many other special numbers as well. Searches on the Internet for any of these special numbers will yield much interesting information about them. You can then explore them further on your own.

The special numbers have fascinated mathematicians for centuries. The branch of mathematics dealing with properties of numbers is called *number theory*. The neat thing about number theory is that no special math background or ability is needed to explore it. You can learn a lot about math just by tinkering with numbers.

A Class Project

Here is an idea for a class project. Make enough slips of paper to have one for each class member. On each slip of paper, write down the name of a special number, repeating entries if needed. Students draw the slips of paper and find out as much as they can about the special number they picked. This can include:

- Why the numbers are special
- How the numbers can be generated step by step
- How the numbers can be found with formulas
- How can you use these numbers to solve math problems
- How these numbers relate to other special numbers, like triangular numbers
- Other features of these numbers

If there are about the same number of slips for each type of number, the students picking the same type of number can work in teams.

The slips of paper can include “wild cards” for making up new special numbers. The students drawing those cards can make up their own special numbers and explore their properties to see what they can find out about them.